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An Extended Inverse Dynamics Control

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Abstract—This letter deals with control problems concerning systems to which an extended inverse control can be applied.

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1. INTRODUCTION

One of the important fields in research on robotics is the applications of the **inverse dynamics control**. In these cases (see, for example, [1] and the references cited there), one considers the dynamic equations of an n-link robot given by

$$\mathbf{M}(\mathbf{q})\,\frac{d^2\mathbf{q}}{dt^2} + \mathbf{h}(\mathbf{q},\,\mathbf{p}) = \mathbf{u}, \tag{1}$$

in which $\mathbf{q} = (q_1,\, q_2, \dots, q_n)^\top$ is a vector of generalized coordinates describing the motion of the system $\mathbf{p} = (\dot{q}_1,\, \dot{q}_2, \dots, \dot{q}_n)^\top$, $\mathbf{M}(\mathbf{q})$ is a given $n \times n$ matrix with $\det \mathbf{M}(\mathbf{q}) > 0$, for all $\mathbf{q} \in \mathcal{D}$, where $\mathcal{D}$ is a given open set in $\Re^n$, $\mathbf{h} : \mathcal{D} \times \mathcal{D} \to \Re^n$ is a given function, and $\mathbf{u} = (u_1, \dots,\, u_n)^\top$ is the vector of the applied torques. Thus, in these cases one chooses a control law of the form

$$\mathbf{u} = \mathbf{M}(\mathbf{q})\,\mathbf{v} + \mathbf{h}(\mathbf{q},\,\mathbf{p}), \tag{2}$$

$\mathbf{v} = (v_1, \dots, v_n)^\top$.

Since $\mathbf{M}$ is invertible on $\mathcal{D}$, equations (1),(2) yield

$$\frac{d^2\mathbf{q}}{dt^2} = \mathbf{v}, \qquad \mathbf{q} \in \mathcal{D}. \tag{3}$$

The control law (2) is called the **inverse dynamics control** (see, for example, [1]). Since the system given by (3) is completely controllable [2], one can use the auxiliary control $\mathbf{v}$ to design the motion of $\{\mathbf{q}(t),\ t \geq 0\}$ within $\mathcal{D}$ and then use (2) to obtain the actual torques $\{\mathbf{u}(t),\ t \geq 0\}$.

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In this letter, we consider dynamic systems which are given by the following equations:

$$\mathbf{N}(\mathbf{q})\,\frac{d^2\mathbf{q}'}{dt^2}+\mathbf{f}(\mathbf{q},\,\mathbf{p})=\mathbf{E}(\mathbf{q}')\,\mathbf{u}', \tag{4}$$

$$\frac{d^2\mathbf{q}''}{dt^2}=\sum_{i=1}^{m}\mathbf{H}_i(\mathbf{q})\,\frac{d^2q_i}{dt^2}+\mathbf{G}(\mathbf{q},\,\mathbf{p}), \tag{5}$$

in which $\mathbf{q}'=(q_1,\,q_2,\ldots,q_m)^\top$, $m\le n$, $\mathbf{N}(\mathbf{q})$ is a given $m\times m$ matrix with $\det\mathbf{N}(\mathbf{q})>0$, $\forall\,\mathbf{q}\in\mathcal{D}$, $\mathbf{E}(\mathbf{q}')$ is a given $m\times m$ matrix with $\det\mathbf{E}(\mathbf{q}')\neq 0$, $\forall\,\mathbf{q}'\in\mathcal{D}'_O$, $\mathcal{D}'_O\subset\Re^m$, $\mathbf{f}$ is a given $\Re^m$ valued vector, and $\mathbf{u}'=(u_1,\ldots,\,u_m)^\top$ is the vector of the applied torques.

In addition, $\mathbf{q}''=(q_{m+1},\ldots,q_n)^\top$ and $\mathbf{H}_i$, $i=1,\ldots,m$ and $\mathbf{G}$ are $\Re^{n-m}$ valued vectors.

Thus, if one chooses a control law of the form

$$\mathbf{E}(\mathbf{q}')\,\mathbf{u}'=\mathbf{N}(\mathbf{q})\,\mathbf{v}'+\mathbf{f}(\mathbf{q},\,\mathbf{p}), \tag{6}$$

where $\mathbf{v}'=(v_1,\ldots,v_m)^\top$, then equations (4) and (6) imply

$$\frac{d^2\mathbf{q}'}{dt^2}=\mathbf{v}',\qquad \mathbf{q}\in\mathcal{D}. \tag{7}$$

Also, (5) and (7) yield

$$\frac{d^2\mathbf{q}''}{dt^2}=\sum_{i=1}^{m}\mathbf{H}_i(\mathbf{q})\,v_i+\mathbf{G}(\mathbf{q},\,\mathbf{p}). \tag{8}$$

The system given by (7) is completely controllable. The goal now is to choose $\mathbf{v}'(\cdot)$ in such a manner that

(i) the trajectories of $\mathbf{q}'$ will be in $\mathcal{D}'_O$,
(ii) the components of $\mathbf{q}''(\cdot)$ will be bounded on $[0,\,\infty)$.

The solution to this problem is out of the scope of this letter and it will be dealt with elsewhere.

Thus, once these specifications are met, then, one can determine, through (6), the values of $\mathbf{u}'$ from the values of $\mathbf{v}'$.

2. EXAMPLE

In this section, a system given by equations (4),(5) is introduced as an example. This example deals with the control of the following dynamical system (see Figure 1). The system is composed of two uniform links of length l and mass m_1 and m_2, respectively. The two links are freely pivoted through their upper ends at a joint $\mathbf{O}$ of mass m_O and coordinates (x,z). That is, $\mathbf{r}_O=x\,\mathbf{I}+z\,\mathbf{K}$, where $\mathbf{I}$, $\mathbf{J}$, and $\mathbf{K}$ are unit vectors along an inertial $(X,\,Y,\,Z)$-coordinate system. The two links are freely pivoted at their lower ends to two identical wheels each of mass m_D and radius a. A torque u_1 is acting on wheel W_1 and a torque u_2 is acting on wheel W_2. A mass M is hanging from $\mathbf{O}$ on a rod of length L, $L<l$. The mass of this rod is negligible with respect to M. In the sequel, it is assumed here that $m_1=m_2=m$. The system motion is confined to the $(X,\,Z)$-plane. It is assumed here that the motion of the wheels on the plane involves rolling without slipping. This leads to the following nonholonomic constraints:

$$\frac{dx}{dt}-l_{O1}\,\frac{d\theta}{dt}\cos\frac{\theta}{2}-a\,\frac{d\psi_1}{dt}=0,\qquad \frac{dx}{dt}+l_{O1}\,\frac{d\theta}{dt}\cos\frac{\theta}{2}-a\,\frac{d\psi_2}{dt}=0, \tag{9}$$

where $l_{O1}=l/2$ and $\frac{d\psi_i}{dt}$ denotes the angular velocity of wheel W_i, $i=1,2$, respectively. Note that equation (9) is integrable. The system described here serves as a simple model to a moving crane.

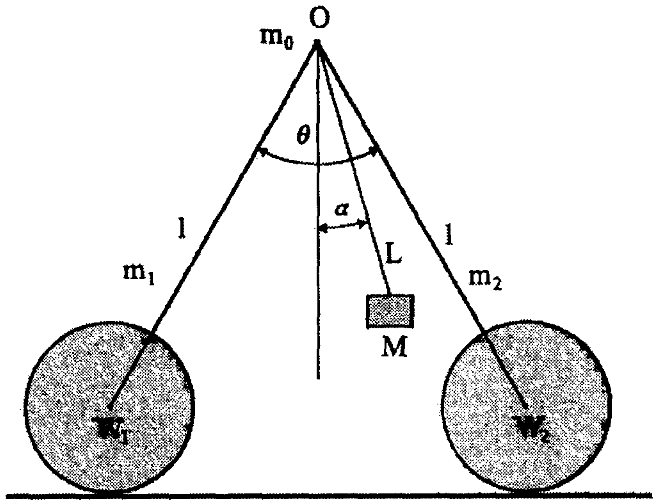

Figure 1. View of the moving crane.

Thus, by applying the Lagrangian method [3], and using some algebraic manipulations, the following equations are obtained:

$$\mathbf{N}(\mathbf{q})\,\frac{d^2\mathbf{q}'}{dt^2}+\mathbf{f}(\mathbf{q},\,\mathbf{p})=\mathbf{E}(\mathbf{q}')\,\mathbf{u}', \tag{10}$$

$$\frac{d^2\mathbf{q}''}{dt^2}=\sum_{i=1}^{2}\mathbf{H}_i(\mathbf{q})\,\frac{d^2q_i}{dt^2}+\mathbf{G}(\mathbf{q},\,\mathbf{p}), \tag{11}$$

where $\mathbf{q}=(x,\,\theta,\,\alpha)^\top$, $\mathbf{q}'=(x,\,\theta)^\top$, $\mathbf{q}''=\alpha$, and denoting the components of $\mathbf{N}$ by N_{ij} and of $\mathbf{E}$ by E_{ij}, $i,j=1,2$, we have

$$N_{11}=I_1+M\sin^2\alpha,\qquad N_{12}=N_{21}=M\,l_{O1}\sin\frac{\theta}{2}\sin\alpha\cos\alpha,$$
$$N_{22}=I_2+m_O\,l_{O1}^2\sin^2\frac{\theta}{2}+M\,l_{O1}^2\sin^2\frac{\theta}{2}\cos^2\alpha+2\,\frac{I_W}{a^2}\,l_{O1}^2\cos^2\frac{\theta}{2},$$
$$I_1=m_O+2\,m+2\,\frac{I_W}{a^2},\qquad I_2=\frac{I_R}{2}+\frac{1}{2}\,m\,l_{O1}^2,\qquad I_W=I_D+m_D\,a^2,$$
$$E_{11}=E_{12}=a^{-1},\qquad E_{21}=-\frac{l_{O1}}{a}\cos\frac{\theta}{2}=-E_{22}.$$

Also, we have

$$\mathbf{H}_1=-\frac{1}{L}\cos\alpha,\qquad \mathbf{H}_2=\frac{l_{O1}}{L}\sin\frac{\theta}{2}\sin\alpha,$$
$$\mathbf{G}=\left[\frac{l_{O1}}{2}\left(\frac{d\theta}{dt}\right)^2\cos\frac{\theta}{2}-g\right]\frac{\sin\alpha}{L}.$$

We denote the components of $\mathbf{f}(\mathbf{q},\,\mathbf{p})$, by f_i, $i=1,2$. These functions are smooth and bounded. However, since the expressions for f_i, $i=1,2$, are not used here explicitly, they are therefore, omitted here to save space.

In the equations above, I_D is the moment of inertia of each of the wheels about its axis and I_R is the moment of inertia of each of the links about a vector in the direction of $\mathbf{J}$ located at the center of mass of the link.

Furthermore, it can be shown that

$$\det\mathbf{N}(\mathbf{q})\geq I_1\,N_{22}(\mathbf{q})\quad\text{and}\quad\det\mathbf{E}(\mathbf{q}')=2\,\frac{l_{O1}}{a^2}\cos\frac{\theta}{2}. \tag{12}$$

Hence, in this example $\mathcal{D} = \Re^3$ and $\mathcal{D}'_O = \Re^2 - (\Re \times \mathcal{L})$, where $\mathcal{L} = \{(2k+1)\pi : k$ $0, \pm 1, \pm 2, \dots\}$.

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